
HOL: Propositional Logic

Overview

- Natural deduction
- Rule application in Isabelle/HOL

Rule notation

$$\frac{A_1 \dots A_n}{A}$$

instead of

$$[[A_1 \dots A_n]] \implies A$$

Natural Deduction

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Introduction: how can I prove $A \oplus B$?

Elimination: what can I prove from $A \oplus B$?

Natural deduction for propositional logic

$$\frac{}{A \wedge B} \text{conjI}$$

$$\frac{}{A \vee B} \quad \frac{}{A \vee B} \text{disjI1/2}$$

$$\frac{}{A \longrightarrow B} \text{impI}$$

$$\frac{}{A = B} \text{iffI}$$

$$\frac{}{\neg A} \text{notI}$$

$$\frac{A \wedge B}{C} \text{conjE}$$

$$\frac{A \vee B}{C} \text{disjE}$$

$$\frac{A \longrightarrow B}{C} \text{impE}$$

$$\frac{}{A \implies B} \text{iffD1} \quad \frac{}{B \implies A} \text{iffD2}$$

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Elimination rule

If I know A_1 and want to prove A
it suffices to prove $A_1 \dots A_n$.

Equality

$$\frac{}{t = t} \text{ refl}$$

$$\frac{s = t}{t = s} \text{ sym}$$

$$\frac{r = s \quad s = t}{r = t} \text{ trans}$$

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Rarely needed explicitly — used implicitly by *simp*

More rules

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Remark:

ccontr and classical are not derivable from the ND-rules.

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Remark:

ccontr and classical are not derivable from the ND-rules.

They make the logic “classical”, i.e. “non-constructive”.

Proof by assumption

$$\frac{A_1 \quad \dots \quad A_n}{A_i} \text{ assumption}$$

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Applying rule $\llbracket A_1; \dots ; A_n \rrbracket \Longrightarrow A$ to subgoal C :

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Example: rule: $\llbracket ?P; ?Q \rrbracket \implies ?P \wedge ?Q$
subgoal: 1. $A \wedge B$

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Result: 1. A
2. B

Rule application: the details

Rule: $\llbracket A_1; \dots ; A_n \rrbracket \implies A$
Subgoal: 1. $\llbracket B_1; \dots ; B_m \rrbracket \implies C$

Rule application: the details

Rule: $\llbracket A_1; \dots ; A_n \rrbracket \Longrightarrow A$
Subgoal: 1. $\llbracket B_1; \dots ; B_m \rrbracket \Longrightarrow C$
Substitution: $\sigma(A) \equiv \sigma(C)$

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Substitution: $\sigma(A) \equiv \sigma(C)$

New subgoals: 1. $\sigma(\llbracket B_1; \dots ; B_m \rrbracket \Longrightarrow A_1)$

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$n.$ $\sigma(\llbracket B_1; \dots ; B_m \rrbracket \Longrightarrow A_n)$

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Command:

apply(rule <rulename>)

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apply assumption

proves

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Demo: proofs by rules

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Like *rule* but also

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New subgoal: 1. $[[X; Y] \Longrightarrow [[A; B] \Longrightarrow Z$

same as: 1. $[[X; Y; A; B] \Longrightarrow Z$

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Demo: elimination rules

Safe and unsafe rules

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Apply safe rules before unsafe ones

\implies **VS** \longrightarrow

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Example: $\llbracket A; B(x) \rrbracket \implies C(x) \rightsquigarrow A \implies B(x) \longrightarrow C(x)$

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Example: $\llbracket A; B(x) \rrbracket \implies C(x) \rightsquigarrow A \implies B(x) \longrightarrow C(x)$

Reverse transformation (after proof):

lemma *abc***[rule_format]**: $A \implies B(x) \longrightarrow C(x)$

Demo: further techniques

HOL: Predicate Logic

Parameters

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1. $\bigwedge x_1 \dots x_n$. *Formula*

The x_i are called **parameters** of the subgoal.

Intuition: local constants, i.e. arbitrary but fixed values.

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Intuition: local constants, i.e. arbitrary but fixed values.

Rules are automatically lifted over $\bigwedge x_1 \dots x_n$ and applied directly to *Formula*.

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- Scope of parameters: whole subgoal
- Scope of \forall , \exists , \dots : ends with ; or \implies

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- Scope of \forall, \exists, \dots : ends with ; or \implies

$$\wedge x y. \llbracket \forall y. P y \longrightarrow Q z y; Q x y \rrbracket \implies \exists x. Q x y$$

means

$$\wedge x y. \llbracket (\forall y_1. P y_1 \longrightarrow Q z y_1); Q x y \rrbracket \implies \exists x_1. Q x_1 y$$

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Bound variables are renamed automatically to avoid name clashes with other variables.

Natural deduction for quantifiers

$\overline{\forall x. P(x)}$ allI

$\frac{\forall x. P(x)}{R}$ allE

$\overline{\exists x. P(x)}$ exI

$\frac{\exists x. P(x)}{R}$ exE

Natural deduction for quantifiers

$$\frac{\wedge x. P(x)}{\forall x. P(x)} \text{allI}$$

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$$\frac{\wedge x. P(x)}{\forall x. P(x)} \text{allI}$$

$$\frac{\forall x. P(x)}{R} \text{allE}$$

$$\frac{P(?x)}{\exists x. P(x)} \text{exI}$$

$$\frac{\exists x. P(x)}{R} \text{exE}$$

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- allI and exE introduce new parameters ($\wedge x$).

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$$\frac{\exists x. P(x) \quad \wedge x. P(x) \implies R}{R} \text{exE}$$

- allI and exE introduce new parameters ($\wedge x$).
- allE and exI introduce new unknowns ($?x$).

Instantiating rules

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Like *rule*, but *?x* in *rule* is instantiated by *term* before application.

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! *x* is in *rule*, not in the goal **!**

Two successful proofs

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apply(rule alll)

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1. $\forall x. \exists y. x = y$

apply(*rule allI*)

1. $\wedge x. \exists y. x = y$

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1. $\wedge x. \exists y. x = y$

best practice

apply(rule_tac x = "x" in exl)

Two successful proofs

1. $\forall x. \exists y. x = y$

apply(rule alll)

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?y \mapsto

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best practice

apply(rule_tac x = "x" in exI)

1. $\wedge x. x = x$

apply(rule refl)

exploration

apply(rule exI)

1. $\wedge x. x = ?y x$

apply(rule refl)

$?y \mapsto \lambda u. u$

Two successful proofs

1. $\forall x. \exists y. x = y$

apply(rule allI)

1. $\wedge x. \exists y. x = y$

best practice

apply(rule_tac x = "x" in exI)

1. $\wedge x. x = x$

apply(rule refl)

simpler & clearer

exploration

apply(rule exI)

1. $\wedge x. x = ?y x$

apply(rule refl)

$?y \mapsto \lambda u. u$

shorter & trickier

Two unsuccessful proofs

$$1. \exists y. \forall x. x = y$$

Two unsuccessful proofs

$$1. \exists y. \forall x. x = y$$

apply(rule_tac x = ??? in exI)

Two unsuccessful proofs

1. $\exists y. \forall x. x = y$

apply(rule_tac x = ??? in ex1)

apply(rule ex1)

1. $\forall x. x = ?y$

Two unsuccessful proofs

1. $\exists y. \forall x. x = y$

apply(rule_tac x = ??? in exI)

apply(rule exI)

1. $\forall x. x = ?y$

apply(rule allI)

1. $\wedge x. x = ?y$

Two unsuccessful proofs

1. $\exists y. \forall x. x = y$

apply(rule_tac x = ??? in ex1)

apply(rule ex1)

1. $\forall x. x = ?y$

apply(rule all1)

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1. $\exists y. \forall x. x = y$

apply(rule_tac x = ??? in ex1)

apply(rule ex1)

1. $\forall x. x = ?y$

apply(rule all1)

1. $\wedge x. x = ?y$

apply(rule refl)

$?y \mapsto x$ yields $\wedge x'. x' = x$

Two unsuccessful proofs

$$1. \exists y. \forall x. x = y$$

apply(rule_tac x = ??? in exI)

apply(rule exI)

$$1. \forall x. x = ?y$$

apply(rule allI)

$$1. \bigwedge x. x = ?y$$

apply(rule refl)

$$?y \mapsto x \text{ yields } \bigwedge x'. x' = x$$

Principle:

**?f $x_1 \dots x_n$ can only be replaced by term t
if $params(t) \subseteq \{x_1, \dots, x_n\}$**

Demo: quantifier proofs

Safe and unsafe rules

Safe allI, exE

Unsafe allE, exI

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Safe allI, exE

Unsafe allE, exI

Create parameters first, unknowns later

Proof methods

Parameter names

Parameter names are chosen by Isabelle

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1. $\forall x. \exists y. x = y$

`apply(rule allI)`

1. $\wedge x. \exists y. x = y$

`apply(rule_tac x = "x" in exI)`

Parameter names

Parameter names are chosen by Isabelle

1. $\forall x. \exists y. x = y$

`apply(rule allI)`

1. $\wedge x. \exists y. x = y$

`apply(rule_tac x = "x" in exI)`

Brittle!

Renaming parameters

1. $\forall x. \exists y. x = y$

`apply(rule allI)`

1. $\wedge x. \exists y. x = y$

`apply(rename_tac xxx)`

1. $\wedge xxx. \exists y. xxx = y$

`apply(rule_tac x = "xxx" in exI)`

Renaming parameters

1. $\forall x. \exists y. x = y$

`apply(rule alll)`

1. $\wedge x. \exists y. x = y$

`apply(rename_tac xxx)`

1. $\wedge xxx. \exists y. xxx = y$

`apply(rule_tac x = "xxx" in exl)`

In general:

`(rename_tac $x_1 \dots x_n$)` renames the rightmost
(inner) n parameters to $x_1 \dots x_n$

Forward proofs: frule and drule

“Forward” rule: $A_1 \Longrightarrow A$

Subgoal: 1. $\llbracket B_1; \dots ; B_n \rrbracket \Longrightarrow C$

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New subgoal: 1. $\sigma(\llbracket B_1; \dots ; B_n; A \rrbracket \Longrightarrow C)$

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Command:

apply(*frule* *rulename*)

Forward proofs: frule and drule

“Forward” rule: $A_1 \Longrightarrow A$

Subgoal: 1. $\llbracket B_1; \dots ; B_n \rrbracket \Longrightarrow C$

Substitution: $\sigma(B_i) \equiv \sigma(A_1)$

New subgoal: 1. $\sigma(\llbracket B_1; \dots ; B_n; A \rrbracket \Longrightarrow C)$

Command:

apply(*frule* rulename)

Like *frule* but also deletes B_i :

apply(*drule* rulename)

frule and drule: the general case

Rule: $\llbracket A_1; \dots ; A_m \rrbracket \Longrightarrow A$

Creates additional subgoals:

$$1. \sigma(\llbracket B_1; \dots ; B_n \rrbracket \Longrightarrow A_2)$$

⋮

$$m-1. \sigma(\llbracket B_1; \dots ; B_n \rrbracket \Longrightarrow A_m)$$

$$m. \sigma(\llbracket B_1; \dots ; B_n; A \rrbracket \Longrightarrow C)$$

Forward proofs: OF

r[OF r₁ ... r_n]

Prove assumption 1 of theorem r with theorem r_1 ,
and assumption 2 with theorem r_2 , and ...

Forward proofs: OF

$r[OF\ r_1\ \dots\ r_n]$

Prove assumption 1 of theorem r with theorem r_1 ,
and assumption 2 with theorem r_2 , and ...

Rule r $\llbracket A_1; \dots ; A_m \rrbracket \implies A$

Rule r_1 $\llbracket B_1; \dots ; B_n \rrbracket \implies B$

Substitution $\sigma(B) \equiv \sigma(A_1)$

$r[OF\ r_1]$

Forward proofs: OF

$$r[OF r_1 \dots r_n]$$

Prove assumption 1 of theorem r with theorem r_1 ,
and assumption 2 with theorem r_2 , and ...

$$\text{Rule } r \quad \llbracket A_1; \dots ; A_m \rrbracket \Longrightarrow A$$

$$\text{Rule } r_1 \quad \llbracket B_1; \dots ; B_n \rrbracket \Longrightarrow B$$

$$\text{Substitution} \quad \sigma(B) \equiv \sigma(A_1)$$

$$r[OF r_1] \quad \sigma(\llbracket B_1; \dots ; B_n; A_2; \dots ; A_m \rrbracket \Longrightarrow A)$$

Forward proofs: THEN

$r_1[THEN\ r_2]$ means $r_2[OF\ r_1]$

Clarifying the goal

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- **apply**(*intro ...*)
Repeated application of intro rules
Example: **apply**(*intro all*)

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- **apply**(*intro ...*)
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Example: **apply**(*intro allI*)
- **apply**(*elim ...*)
Repeated application of elim rules
Example: **apply**(*elim conjE*)

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Repeated application of intro rules
Example: **apply(*intro all*)**
- **apply(*elim ...*)**
Repeated application of elim rules
Example: **apply(*elim conjE*)**
- **apply(*clarify*)**
Repeated application of safe rules
without splitting the goal

Clarifying the goal

- **apply**(*intro ...*)
Repeated application of intro rules
Example: **apply**(*intro all*)
- **apply**(*elim ...*)
Repeated application of elim rules
Example: **apply**(*elim conjE*)
- **apply**(*clarify*)
Repeated application of safe rules
without splitting the goal
- **apply**(*clarsimp simp add: ...*)
Combination of *clarify* and *simp*.