Cardinals in Higher-Order Logic

Jasmin Blanchette, Andrei Popescu and Dmitriy Traytel

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Overview

- Motivation
- Confession
- Formalization

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Motivation

What happens if we add impredicative polymorphism to HOL?

$$\frac{\Gamma, \alpha : \mathsf{Type} \vdash \sigma : \mathsf{Type}}{\Gamma \vdash \prod \alpha.\, \sigma \ : \ \mathsf{Type}}$$

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- Start with a functor $\alpha\,\mathsf{F}$ that has no fixpoint,
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$\mathsf{Fmap} : (\alpha \to \beta) \to \alpha\ \mathsf{F} \to \beta\ \mathsf{F}$

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$$\begin{aligned}&\text{mor}\ (A, s)\ (B, t)\ h \equiv \\ &\quad \text{image}\ h\ A \subseteq B\ \wedge \\ &\quad (\forall x \in \text{F}\ A.\ t\ (\text{Fmap}\ h\ x) = h\ (s\ x))\end{aligned}$$

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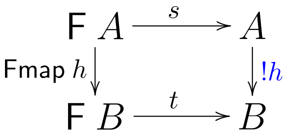

quasi initial if, for all algebras (B, t),
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initial if it is both quasi initial and weakly initial

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Lambek's Lemma: For any initial algebra (A, s), the function s is a bijection between $\mathsf{F}\ A$ and A.

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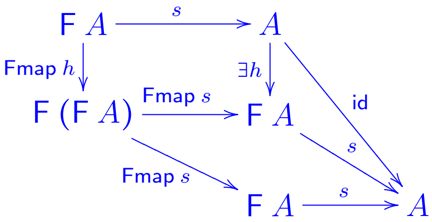
Proof.



1. Obtain h from weak initiality of (A, s)
2. $s \circ h$ = id from quasi initiality of (A, s)
3. $h \circ s$ = Fmap $s \circ$ Fmap h = Fmap $(s \circ h)$ = Fmap id = id
4. From 2 and 3: h is the inverse of s

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Lemma: Any algebra (A, s) has a subalgebra that is quasi initial, namely its minimal subalgebra $\mathsf{minSub}\,(A, s) = (A_0, s)$ where $A_0 \equiv \bigcap \{B \subseteq A \mid \mathsf{alg}\,(B, s)\}$.

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Proof. Immediate by fixpoint induction, noting that $A_0 = \mathsf{lfp}\ (\lambda B.\ \mathsf{image}\ s\ (\mathsf{F}\ B))$.

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Corollary: The minimal subalgebra of any weakly initial algebra (A, s) is an initial algebra.

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Proof. Uniqueness from the lemma.

Existence from weak initiality of (A, s):

$$\begin{array}{ccc}
\mathsf{F}\ A_0 & \xrightarrow{s} & A_0 \\
{\scriptstyle \mathsf{Fmap\ id}}\downarrow & & \downarrow{\scriptstyle \mathsf{id}} \\
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$$P_0 \equiv \prod \alpha. ((\mathsf{unit} + \alpha) \to \alpha) \to \alpha$$

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Coquand (1994): Internalized in impredicative HOL

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Cardinals in Higher-Order Logic

with Application to Modular (Co)datatypes

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Yes, if Fatms x is bounded by a fixed cardinal Fbd.

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Then $|A_0| \leq \mathsf{Fbd}$.

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Lemma: Assume Fbd regular cardinal, and
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Then $|A_0| \leq \mathsf{Fbd}$.

Proof. Recall that
$A_0 = \bigcap\{B \subseteq A \mid \mathsf{alg}\ (B, s)\} = \mathsf{lfp}\ (\lambda B.\ \mathsf{image}\ s\ (\mathsf{F}\ B))$.
Need alternative definition "from below": $A_0 = \bigcup_{i \text{ cardinal}} B_i$

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Hence, by cardinal arithmetic, $|A_0| \leq \mathsf{Fbd}$.
QED

Try the proof in polymorphic HOL without impredicativity

1. Build a weakly initial algebra
as the product (P_0, s_0) of all algebras

$$P_0 \equiv \prod \alpha. \prod (A, s) \in \mathsf{Algs}_\alpha.\ A$$

where $\mathsf{Algs}_\alpha = \{(A, s) \in \alpha\ \mathsf{set} \times (\alpha\ \mathsf{F} \to \alpha) \mid \mathsf{alg}\ (A, s)\}$

$$s_0\ x \equiv \lambda(A, s).\ s\ (\mathsf{Fmap}\ (\lambda p.\ p\ (A, s))\ x)$$ ✓

2. Take (I_0, s_0) to be the minimal subalgebra of (P_0, s_0). ✓

3. By Lambek, s_0 is a bijection between I_0 and F I_0,
contradicting our assumption about F. ✓

Try the proof in polymorphic HOL without impredicativity

1. Build a weakly initial algebra
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Fatms no longer bounded!

Outcome

Failed to prove inconsistency of predicative HOL ☺

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But did construct the initial algebra abstractly for $\alpha\,\mathsf{F}$

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But did construct the initial algebra abstractly for $\alpha\,\mathsf{F}$
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Outcome

Failed to prove inconsistency of predicative HOL ☺

But did construct the initial algebra abstractly for $\alpha\,\mathsf{F}$

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$$\Downarrow$$

Bounded Natural Functor (BNF)

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Bounded Natural Functor (BNF)

$\Downarrow$

Modular, Open-Ended (Co)datatypes in Isabelle/HOL

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Modular, Open-Ended (Co)datatypes in Isabelle/HOL

(dual construction yields final coalgebra)

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without impredicativity, also need boundedness (Fbd)

$$\Downarrow$$

Bounded Natural Functor (BNF)

$$\Downarrow$$

Modular, Open-Ended (Co)datatypes in Isabelle/HOL

Outcome

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But did construct the initial algebra abstractly for α F

with impredicativity, suffices natural functor (Fatms, Fmap)

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Bounded Natural Functor (BNF)

$\Downarrow$

Modular, Open-Ended (Co)datatypes in Isabelle/HOL

datatype α list = Nil | Cons α (α list)

Outcome

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Modular, Open-Ended (Co)datatypes in Isabelle/HOL

datatype α list = Nil | Cons α (α list)

α list = lfp ($\lambda\beta.$ unit + $\alpha \times \beta$)

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Bounded Natural Functor (BNF)

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Modular, Open-Ended (Co)datatypes in Isabelle/HOL

datatype α list = Nil | Cons α (α list)
α list = lfp ($\lambda\beta$. unit + $\alpha \times \beta$)
codatatype α tree = Node α (α tree list)

Outcome

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α tree = gfp ($\lambda\beta$. $\alpha \times \beta$ list)

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codatatype α tree = Node α (α tree bag)

α tree = gfp ($\lambda\beta$. $\alpha \times \beta$ list)

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datatype α list = Nil | Cons α (α list)

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codatatype α tree = Node α (α tree ?) – you name it

α tree = gfp ($\lambda\beta$. $\alpha \times \beta$ list)

Formalization

On the way, formalized rich theory of ordinals and cardinals

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- OK for our (co)datatype constructions

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- OK for our (co)datatype constructions
- not OK for proving fancier results about cardinals

Related Work

Related Work

- Paulson and Grabczewski (1996) in Isabelle/ZF: Ordinals and Cardinals
- Harrison in HOL Light: Cardinality Reasoning
- Huffman (2004) in Isabelle/HOL: Countable Ordinals
- Norrish and Huffman (2014) in HOL4: Ordinals

Conclusion

Impredicative polymorphism is not set-theoretic (Reynolds) hence inconsistent in HOL (Coquand)

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Impredicative polymorphism is not set-theoretic (Reynolds) hence inconsistent in HOL (Coquand)

However

Predicative polymorphism + formalized cardinals are fruitfully category-theoretic in HOL
... and probably even more so in HOL_ω, Coq, etc.

Cardinals in Higher-Order Logic with Application to Modular (Co)Datatypes

Jasmin Blanchette, Andrei Popescu and Dmitriy Traytel

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