

Nominal Techniques Course

Thursday-Lecture

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Permutations on Fun's

$$\pi \bullet fn \stackrel{\text{def}}{=} \lambda x. \pi \bullet (fn(\pi^{-1} \bullet x))$$

Example $\lambda x. pr(a, x)$:

Permutations on Fun's

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Example $\lambda x. pr(a, x)$: What is this function?

$a \mapsto pr(a, a)$
 $b \mapsto pr(a, b)$
 $c \mapsto pr(a, c)$
 $d \mapsto pr(a, d)$
 \vdots

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Example $\lambda x. pr(a, x)$: What is this function?

$$(a \ b) \bullet \begin{array}{l} a \mapsto pr(a, a) \\ b \mapsto pr(a, b) \\ c \mapsto pr(a, c) \\ d \mapsto pr(a, d) \\ \vdots \end{array}$$

Permutations on Fun's

$$\pi \cdot fn \stackrel{\text{def}}{=} \lambda x. \pi \cdot (fn(\pi^{-1} \cdot x))$$

Example $\lambda x. pr(a, x)$: What is this function?

$$(a \ b) \cdot \begin{array}{l} a \mapsto pr(a, a) \\ b \mapsto pr(a, b) \\ c \mapsto pr(a, c) \\ d \mapsto pr(a, d) \\ \vdots \end{array}$$

Permutations on Fun's

$$\pi \bullet fn \stackrel{\text{def}}{=} \lambda x. \pi \bullet (fn(\pi^{-1} \bullet x))$$

Example $\lambda x. pr(a, x)$: What is this function?

$$\begin{aligned} b &\mapsto pr(b, b) \\ a &\mapsto pr(b, a) \\ c &\mapsto pr(b, c) \\ d &\mapsto pr(b, d) \\ &\vdots \end{aligned}$$

which is the function $\lambda x. pr(b, x)$.

Permutations on Fun's (ct.)

$$\pi \bullet fn \stackrel{\text{def}}{=} \lambda x. \pi \bullet (fn(\pi^{-1} \bullet x))$$

So $(a\ b) \bullet \lambda x. pr(a, x) = \lambda x. pr(b, x)$!

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$$\begin{aligned} & (a\ b) \bullet \lambda x. pr(a, x) \\ = & \lambda y. (a\ b) \bullet ((\lambda x. pr(a, x))((a\ b) \bullet y)) \end{aligned}$$

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Equality on Functions

The question arose whether ($a \neq b$):

$$[a].[a].\text{am}(a) = [b].[a].\text{am}(a)?$$

Well, if we knew

$$\blacksquare \pi \cdot ([a].t) = [\pi \cdot a].(\pi \cdot t)$$

$$\blacksquare t_1 = t_2 \Leftrightarrow [a].t_1 = [a].t_2$$

$$\blacksquare a \neq b \Rightarrow (t_1 = (a \ b) \cdot t_2 \wedge a \neq t_2 \\ \Leftrightarrow [a].t_1 = [b].t_2)$$

we could easily decide this question, namely:

Equality on Functions (ct.)

$$[a].[a].\text{am}(a) = [b].[a].\text{am}(a) \text{ where } a \neq b$$

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iff $[a].\text{am}(a) = (a \ b) \cdot [a].\text{am}(a)$

and $a \neq [a].\text{am}(a)$

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Proofs for the Equalities 1

$$\pi \bullet ([a].t) = [\pi \bullet a].(\pi \bullet t)$$

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$\lambda x. \pi \bullet \text{if } \pi^{-1} \bullet x = a \text{ then } t$
 $\quad \text{else if } \pi^{-1} \bullet x \neq t \text{ then } (a \ \pi^{-1} \bullet x) \bullet t \text{ else er}$

$= \lambda x. \text{if } x = \pi \bullet a \text{ then } \pi \bullet t$
 $\quad \text{else if } x \neq \pi \bullet t \text{ then } (\pi \bullet a \ x) \bullet \pi \bullet t \text{ else er}$

$[a].t \stackrel{\text{def}}{=} \lambda x. \text{if } x = a \text{ then } t$
 $\quad \text{else if } x \neq t \text{ then } (x \ a) \bullet t \text{ else er}$

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$$\pi \bullet \text{if } \dots \text{ then } \dots \text{ else } \dots = \\ \text{if } \dots \text{ then } \pi \bullet \dots \text{ else } \pi \bullet \dots$$

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$$\pi \bullet ([a].t) = [\pi \bullet a].(\pi \bullet t)$$

which is:

$\lambda x.$ if $\pi^{-1} \bullet x = a$ then $\pi \bullet t$
else if $\pi^{-1} \bullet x \neq t$ then $\pi \bullet (a \ \pi^{-1} \bullet x) \bullet t$ else er

$= \lambda x.$ if $x = \pi \bullet a$ then $\pi \bullet t$
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$$\begin{aligned} & \pi \bullet (a \ \pi^{-1} \bullet x) \bullet t \\ &= (\pi \bullet a \ \pi \bullet \pi^{-1} \bullet x) \bullet \pi \bullet t \\ &= (\pi \bullet a \ x) \bullet \pi \bullet t \end{aligned}$$

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$$\pi^{-1} \bullet x \neq t \text{ iff } x \neq \pi \bullet t$$

Proofs for the Equalities 1

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Done.

Proofs for the Equalities 2

$$[a].t_1 = [a].t_2 \Rightarrow t_1 = t_2$$

which means we can assume that:

$$\begin{aligned} & \lambda x. \text{if } x = a \text{ then } t_1 \\ & \quad \text{else if } x \neq t_1 \text{ then } (a \ x) \bullet t_1 \text{ else er} \\ = & \lambda x. \text{if } x = a \text{ then } t_2 \\ & \quad \text{else if } x \neq t_2 \text{ then } (a \ x) \bullet t_2 \text{ else er} \end{aligned}$$

Proofs for the Equalities 2

$$[a].t_1 = [a].t_2 \Rightarrow t_1 = t_2$$

which means we can assume that:

$\forall x.$ if $x = a$ then t_1
else if $x \neq t_1$ then $(a \ x) \bullet t_1$ else er
=
if $x = a$ then t_2
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$$[a].t_1 = [a].t_2 \Rightarrow t_1 = t_2$$

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else if $a \neq t_1$ then $(a \ a) \bullet t_1$ else er
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Proofs for the Equalities 2

$$[a].t_1 = [a].t_2 \Rightarrow t_1 = t_2$$

which means we can assume that:

t_1

=

t_2

Done.

Freshness 2

Lemma: $a \neq b \wedge b \# [a].t \Rightarrow b \# t$

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(1) $(\exists c)c \# (a, b, t, [a].t)$

"finitely supported"

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(2) $(b\ c) \bullet [a].t = [a].t$

from (1) + ass.

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by “same abstraction”

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(4) $(b\ c) \bullet c \# (b\ c) \bullet t$

from $c \# t$

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(3) $(b\ c) \bullet t = t$

by “same abstraction”

(4) $b \# t$

from $c \# t$ and (3)

Freshness 2

Lemma: $a \neq b \wedge b \# [a].t \Rightarrow b \# t$

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“finitely supported”

(2) $[a].((b\ c) \bullet t) = [a].t$

from (1) + ass

(3) $(b\ c) \bullet t = t$

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(4) $b \# t$

from $c \# t$ and (3)

Done.

Freshness 3

Lemma: $a \# [a].t$

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Proof:

(1) $(\exists c)c \# (a, t)$

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Freshness 3

Lemma: $a \# [a].t$

Proof:

(1) $(\exists c)c \# (a, t)$

(2) $c \# [a].t$

“finitely supported”
by (Freshness 1)

Freshness 3

Lemma: $a \# [a].t$

Proof:

(1) $(\exists c)c \# (a, t)$

(2) $c \# [a].t$

(3) $(a\ c) \bullet c \# (a\ c) \bullet [a].t$

“finitely supported”

by (Freshness 1)

from (2)

Freshness 3

Lemma: $a \# [a].t$

Proof:

(1) $(\exists c)c \# (a, t)$

(2) $c \# [a].t$

(3) $a \# [c].((a\ c) \bullet t)$

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Freshness 3

Lemma: $a \# [a].t$

Proof:

$$(1) (\exists c)c \# (a, t)$$

$$(2) c \# [a].t$$

$$(3) a \# [c].((a c) \bullet t)$$

$$(4) [c].((a c) \bullet t) = [a].t$$

"finitely supported"

by (Freshness 1)

from (2)

provided $c \# t$ and
 $(a c) \bullet t = (a c) \bullet t$

Freshness 3

Lemma: $a \# [a].t$

Proof:

$$(1) (\exists c)c \# (a, t)$$

$$(2) c \# [a].t$$

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$$(4) [c].((a c) \bullet t) = [a].t$$

“finitely supported”

by (Freshness 1)

from (2)

provided $c \# t$ and
 $(a c) \bullet t = (a c) \bullet t$

Both hold, therefore $a \# [a].t$

Done.

Equivariance

$\text{equiv}(P) \stackrel{\text{def}}{=} (\forall t \cdot \Lambda) (\forall r \cdot \text{ESTime}) (\forall ni)$

We have an induction principle for Λ_α (by way of how it is constructed), but this induction principle is not as weak (in the good sense) as that from Nominal Logic. If we prove an equivariant property, we can do better.

slightly unusual definition for equivariance.

Equivariance

$$\text{eqvt}(P) \stackrel{\text{def}}{=} (\forall t : \Lambda_a) (\forall x : \mathit{FSType}) (\forall pi) \\ P t x \Rightarrow P(\pi \bullet t)(\pi \bullet x)$$

Later we shall often consider predicates having an Λ_α -term as first argument and an FSType as second argument. Therefore, this slightly unusual definition for equivariance.

Some /Any-Property

Assuming $eqvt(P)$ then

$$(\exists x) a \# x \wedge (\forall t) P([a].t) x$$

if and only if

$$(\forall x) a \# x \Rightarrow (\forall t) P([a].t) x$$

Some /Any-Property

Assuming $eqvt(P)$ then

$$(\exists x) a \# x \wedge (\forall t) P([a].t) x$$

if and only if

$$(\forall x) a \# x \Rightarrow (\forall t) P([a].t) x$$

Proof: Same as on Tuesday.

Induction

$$(\forall a) P (\text{am}(a)) x$$

$$(\forall t_1, t_2) P t_1 x \wedge P t_2 x \Rightarrow P (\text{pr}(t_1, t_2)) x$$

$$(\exists a) a \neq x \wedge (\forall t) P t x \Rightarrow P ([a].t) x$$

$$(\forall t) P t x$$

Proof: By induction on n (the "stage" when constructing Λ_α).

Induction

$eqvt(P)$

$(\forall a) P(am(a)) x$

$(\forall t_1, t_2) P t_1 x \wedge P t_2 x \Rightarrow P(pr(t_1, t_2)) x$

$(\forall a) a \# x \Rightarrow (\forall t) P t x \Rightarrow P([a].t) x$

$(\forall t) P t x$

Proof: By induction on n (the "stage" when constructing Λ_α).

What Has Been Achieved

- we gave an inductive definition of a set (Λ_α) that is bijective with the α -equated lambda-terms
- Λ_α has very much the feel of (named) lambda-terms (equated up to α -equivalence)
- if we can prove equivariance for the IH, then we only need to prove the abstraction case for one fresh atom
- and we can put the money where our mouth is...:o)