Semantics
of Programming Languages (24)

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Task for Today

Assuming that a and b are distinct variables, is it possible to find λ-terms M_1 to M_7 that make the following pairs α-equivalent?

- $\lambda a.\lambda b.(M_1\ b)$ and $\lambda b.\lambda a.(a\ M_1)$
- $\lambda a.\lambda b.(M_2\ b)$ and $\lambda b.\lambda a.(a\ M_3)$
- $\lambda a.\lambda b.(b\ M_4)$ and $\lambda b.\lambda a.(a\ M_5)$
- $\lambda a.\lambda b.(b\ M_6)$ and $\lambda a.\lambda a.(a\ M_7)$

If there is one solution for a pair, can you describe all its solutions?

Typing Rules

$$\frac{(x\!:\!\tau) \in \Gamma \quad \text{valid}\,\Gamma}{\Gamma \vdash x : \tau}$$

$$\frac{\Gamma \vdash t_1 : \sigma \to \tau \quad \Gamma \vdash t_2 : \sigma}{\Gamma \vdash t_1\, t_2 : \tau}$$

$$\frac{x \# \Gamma \quad (x\!:\!\sigma)\!::\!\Gamma \vdash t : \tau}{\Gamma \vdash \lambda x.t : \sigma \to \tau}$$

First-Order Unification

- Terms:

  variables: X
  function symbols: $F(t_1, \ldots, t_n)$

- Substitutions:

  $$[X_1 := t_1, \ldots, X_n := t_n]$$

- Unification problems:

  $$\{t_1 \approx ?\ u_1, \ldots, t_n \approx ?\ u_n\}$$

- Solutions: is a substitution σ such that

  $$\sigma(t_1) = \sigma(u_1) \ldots \sigma(t_n) = \sigma(u_n)$$

Examples

- $F(X) \approx ?\ F(Y)$
- $F(X) \approx ?\ G(Y)$
- $F(G(X)) \approx ?\ F(Y)$
- $H(G(X), X) \approx ?\ H(Y, t)$
- $X \approx ?\ F(X)$

Unification Algorithm

- $\mathtt{f}(t_1, \ldots, t_n) \approx ? \; \mathtt{f}(s_1, \ldots, s_n) \cup P$
  $\Longrightarrow t_1 \approx ? \; s_1 \cup \ldots \cup t_n \approx ? \; s_n \cup P$
- $X \approx ? \; t \cup P$
  $\stackrel{[X:=t]}{\Longrightarrow} P[X := t]$ $\qquad$ $[X := t]$ part of the answer
- ...
- if σ is an answer for a problem P, then σ is the most general solution

MGU

- σ is a **most general unifier** for a problem P if, for every other solution σ' for P, there exists a substitution δ such that

$$\delta \circ \sigma = \sigma'$$

Recap α-Equivalence

$$\frac{}{a \approx a}\ \approx\text{-atm} \qquad \frac{t_1 \approx s_1 \quad t_2 \approx s_2}{t_1\, t_2 \approx s_1\, s_2}\ \approx\text{-app}$$

$$\frac{t \approx s}{\lambda a.t \approx \lambda a.s}\ \approx\text{-lam}_1 \qquad \frac{t \approx (a\, b)\bullet s \quad a \# s}{\lambda a.t \approx \lambda b.s}\ \approx\text{-lam}_2$$

$$\frac{}{a \# b}\ \#\text{-atm} \qquad \frac{a \# t_1 \quad a \# t_2}{a \# t_1\, t_2}\ \#\text{-app}$$

$$\frac{}{a \# \lambda a.t}\ \#\text{-lam}_1 \qquad \frac{a \# t}{a \# \lambda b.t}\ \#\text{-lam}_2$$

assuming $a \neq b$

Terms

- $\langle\rangle$ Units
- $\langle t, t' \rangle$ Pairs
- $F\, t$ Funct.

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- $\langle t, t' \rangle$ Pairs
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- a Atoms
- $a.t$ Abstractions

bindable names (of object-level variables etc.)

generic binder:

$\ulcorner \lambda a.a \urcorner \mapsto \mathtt{fn}\ a.a$

constructions like $\mathtt{fn}\ X.X$ are not allowed

Terms

- $\langle\rangle$ Units
- $\langle t, t' \rangle$ Pairs
- $F\,t$ Funct.
- a Atoms
- $a.t$ Abstractions
- $\pi \cdot X$ Suspensions

Terms

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- $\pi \cdot X$ Suspensions

π is an explicit permutation, which is a list of swappings $(a_1\, b_1) \ldots (a_n\, b_n)$, waiting to be applied to the term that is substituted for X

X is a variable standing for an unknown term

Permutations

a permutation applied to a term:

- $$[]\bullet a \stackrel{\text{def}}{=} a$$
- $$(b\,c)::\pi\bullet a \stackrel{\text{def}}{=} \begin{cases} c & \text{if } \pi\bullet a = b \\ b & \text{if } \pi\bullet a = c \\ \pi\bullet a & \text{otherwise} \end{cases}$$

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- $$\pi \bullet a.t \stackrel{\text{def}}{=} \pi \bullet a.\pi \bullet t$$

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- $$\pi \bullet a.t \stackrel{\text{def}}{=} \pi \bullet a.\pi \bullet t$$
- $$\pi \bullet \pi' \cdot X \stackrel{\text{def}}{=} (\pi @ \pi') \cdot X$$

Permutations

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- $$\pi\bullet a.t \stackrel{\text{def}}{=} \pi\bullet a.\pi\bullet t$$
- $$\pi\bullet\pi'\cdot X \stackrel{\text{def}}{=} (\pi @ \pi')\cdot X$$

inverse permutation given by reversing the list of swappings

Permutations on atoms are bijections!

$$\pi\bullet a = b \quad \text{iff} \quad a = (\pi^{-1})\bullet b$$

Freshness Relation

We will identify

$$\mathtt{fn}\, a.X \;\approx\; \mathtt{fn}\, b.(a\, b)\cdot X$$

provided that 'b is fresh for X — $(b \# X)$',
i.e., does not occur freely in any ground term
that might be substituted for X.

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explicit permutation — waits to be applied to the term that is substituted for X

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$$\mathtt{fn}\, \boldsymbol{a}.\boldsymbol{X} \;\approx\; \mathtt{fn}\, \boldsymbol{b}.(\boldsymbol{a}\,\boldsymbol{b})\cdot\boldsymbol{X}$$

provided that '$\boldsymbol{b}$ is fresh for $\boldsymbol{X}$ — ($\boldsymbol{b} \# \boldsymbol{X}$)', i.e., does not occur freely in any ground term that might be substituted for $\boldsymbol{X}$.

Freshness Relation

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provided that 'b is fresh for X — (b # X)', i.e., does not occur freely in any ground term that might be substituted for X.

If we know more about X, e.g., if we knew that $a \;\#\; X$ and $b \;\#\; X$, then we can replace $(a\, b)\cdot X$ by X.

Freshness Assumptions

Our equality is **not** just

$$t \approx t'$$ α-equivalence

Freshness Assumptions

but judgements

$$\nabla \vdash t \approx t' \qquad \alpha\text{-equivalence}$$

where

$$\nabla = \{a_1 \# X_1, \ldots, a_n \# X_n\}$$

is a finite set of freshness assumptions.

Freshness Assumptions

but judgements

$$\nabla \vdash t \approx t' \qquad \alpha\text{-equivalence}$$

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$$\{a \# X, b \# X\} \;\vdash\; \mathtt{fn}\, a.X \approx \mathtt{fn}\, b.X$$

Freshness Assumptions

but judgements

$\nabla \vdash t \approx t'$ α-equivalence

$\nabla \vdash a \# t$ freshness

where

$$\nabla = \{a_1 \# X_1, \ldots, a_n \# X_n\}$$

is a finite set of freshness assumptions.

$$\begin{array}{rcl} \{b \# X\} & \vdash & b \# a.X \\ \{\} & \vdash & a \# a.X \end{array}$$

Rules for Equivalence

Excerpt
(i.e. only the interesting rules)

Rules for Equivalence

$$\frac{\nabla \vdash t \approx t'}{\nabla \vdash a.t \approx a.t'}$$

$$\frac{a \neq b \qquad \nabla \vdash t \approx (a\,b)\bullet t' \qquad \nabla \vdash a \,\#\, t'}{\nabla \vdash a.t \approx b.t'}$$

Rules for Equivalence

$$\frac{(a \;\#\; X) \in \nabla \quad \text{for all } a \text{ with } \pi \bullet a \neq \pi' \bullet a}{\nabla \vdash \pi {\cdot} X \approx \pi' {\cdot} X}$$

Rules for Equivalence

$$\frac{(a \# X) \in \nabla \quad \text{for all } a \text{ with } \pi \bullet a \neq \pi' \bullet a}{\nabla \vdash \pi \cdot X \approx \pi' \cdot X}$$

for example

$$\{a \# X, b \# X\} \vdash X \approx (a\, b) \cdot X$$

Rules for Equivalence

$$\frac{(a \# X) \in \nabla \quad \text{for all } a \text{ with } \pi \bullet a \neq \pi' \bullet a}{\nabla \vdash \pi \cdot X \approx \pi' \cdot X}$$

for example

$$\{a \# X, c \# X\} \vdash (a\,c)(a\,b) \cdot X \approx (b\,c) \cdot X$$

because $(a\,c)(a\,b)$: $a \mapsto b$, $b \mapsto c$, $c \mapsto a$ $\quad (b\,c)$: $a \mapsto a$, $b \mapsto c$, $c \mapsto b$

disagree at a and c.

Rules for Freshness

Excerpt
(again only the interesting rules)

Rules for Freshness

$$\frac{a \neq b}{\nabla \vdash a \# b}$$

$$\frac{}{\nabla \vdash a \# a.t} \qquad \frac{a \neq b \quad \nabla \vdash a \# t}{\nabla \vdash a \# b.t}$$

$$\frac{(\pi^{-1} \bullet a \# X) \in \nabla}{\nabla \vdash a \# \pi \cdot X}$$

$\approx$ is an Equivalence

Theorem: $\approx$ is an equivalence relation.

(Reflexivity)	$\nabla \vdash t \approx t$
(Symmetry)	if $\nabla \vdash t_1 \approx t_2$ then $\nabla \vdash t_2 \approx t_1$
(Transitivity)	if $\nabla \vdash t_1 \approx t_2$ and $\nabla \vdash t_2 \approx t_3$ then $\nabla \vdash t_1 \approx t_3$

$\approx$ is an Equivalence

Theorem: $\approx$ is an equivalence relation.

because $\approx$ has very good properties:

- $\nabla \vdash t \approx t'$ then $\nabla \vdash \pi \bullet t \approx \pi \bullet t'$
- $\nabla \vdash a \# t$ then $\nabla \vdash \pi \bullet a \# \pi \bullet t$

$\approx$ is an Equivalence

Theorem: $\approx$ is an equivalence relation.

because $\approx$ has very good properties:

- $\nabla \vdash t \approx t'$ then $\nabla \vdash \pi \bullet t \approx \pi \bullet t'$
- $\nabla \vdash a \mathrel{\#} t$ then $\nabla \vdash \pi \bullet a \mathrel{\#} \pi \bullet t$
- $\nabla \vdash t \approx \pi \bullet t'$ then $\nabla \vdash (\pi^{-1}) \bullet t \approx t'$
- $\nabla \vdash a \mathrel{\#} \pi \bullet t$ then $\nabla \vdash (\pi^{-1}) \bullet a \mathrel{\#} t$

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because $\approx$ has very good properties:

- $\nabla \vdash t \approx t'$ then $\nabla \vdash \pi{\bullet}t \approx \pi{\bullet}t'$
- $\nabla \vdash a \mathrel{\#} t$ then $\nabla \vdash \pi{\bullet}a \mathrel{\#} \pi{\bullet}t$
- $\nabla \vdash t \approx \pi{\bullet}t'$ then $\nabla \vdash (\pi^{-1}){\bullet}t \approx t'$
- $\nabla \vdash a \mathrel{\#} \pi{\bullet}t$ then $\nabla \vdash (\pi^{-1}){\bullet}a \mathrel{\#} t$
- $\nabla \vdash a \mathrel{\#} t$ and $\nabla \vdash t \approx t'$ then $\nabla \vdash a \mathrel{\#} t'$

Substitutions

- $\sigma(a.t) \stackrel{\text{def}}{=} a.\sigma(t)$
- $\sigma(\pi{\cdot}X) \stackrel{\text{def}}{=} \begin{cases} \pi \bullet \sigma(X) & \text{if } \sigma(X) \neq X \\ \pi{\cdot}X & \text{o'wise do nothing} \end{cases}$

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for example

$$a.(a\,b){\cdot}X\ \ [X := \langle b, Y\rangle]$$

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for example

$$\underline{a.(a\,b){\cdot}X\ \ [X := \langle b, Y\rangle]}$$

$$\Rightarrow \quad a.\underline{(a\,b){\cdot}X[X := \langle b, Y\rangle]}$$

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for example

$$a.(a\,b){\cdot}X\ \ [X := \langle b, Y\rangle]$$

$$\Rightarrow\quad a.\underline{(a\,b){\cdot}X[X := \langle b, Y\rangle]}$$

$$\Rightarrow\quad a.(a\,b)\bullet\underline{\langle b, Y\rangle}$$

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for example

$$a.(a\,b)\cdot X \;\; [X := \langle b, Y \rangle]$$

$$\Rightarrow \quad a.(a\,b)\cdot X[X := \langle b, Y \rangle]$$

$$\Rightarrow \quad a.\underline{(a\,b)}\bullet \langle b, Y \rangle$$

$$\Rightarrow \quad a.\langle a, (a\,b)\cdot Y \rangle$$

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- if $\nabla \vdash t \approx t'$ and $\nabla' \vdash \sigma(\nabla)$

  then $\nabla' \vdash \sigma(t) \approx \sigma(t')$

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this means
$\nabla' \vdash a \,\#\, \sigma(X)$
holds for all
$(a \,\#\, X) \in \nabla$

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- if $\nabla \vdash t \approx t'$ and $\nabla' \vdash \sigma(\nabla)$

  then $\nabla' \vdash \sigma(t) \approx \sigma(t')$
- $\sigma(\pi \bullet t) = \pi \bullet \sigma(t)$

Equational Problems

An equational problem

$$t \approx ?\ t'$$

is **solved** by

- a substitution σ (terms for variables)
- and a set of freshness assumptions ∇

so that $\nabla \vdash \sigma(t) \approx \sigma(t')$.

Unifying equations may entail solving freshness problems.

E.g. assuming that $a \neq a'$, then

$$a.t \approx ?\ a'.t'$$

can only be solved if

$$t \approx ?\ (a\,a')\bullet t' \quad \text{and} \quad a \;\#?\; t'$$

can be solved.

Freshness Problems

A freshness problem

$$a \;\#?\; t$$

is solved by

- a substitution σ
- and a set of freshness assumptions ∇

so that $\nabla \vdash a \;\#\; \sigma(t)$.

Most General Unifiers

Definition: for a unification problem P, a solution (σ_1, ∇_1) is more general than another solution (σ_2, ∇_2), iff there exists a substitution δ with

- $\nabla_2 \vdash \delta(\nabla_1)$
- $\nabla_2 \vdash \sigma_2 \approx \delta \circ \sigma_1$

Most General Unifiers

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- $\nabla_2 \vdash \delta(\nabla_1)$
- $\nabla_2 \vdash \sigma_2 \approx \delta \circ \sigma_1$

$\nabla_2 \vdash a \# \delta(X)$ holds for all $(a \# X) \in \nabla_1$

Most General Unifiers

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$\nabla_2 \vdash \sigma_2(X) \approx \delta(\sigma_1(X))$
holds for all
$X \in \mathrm{dom}(\sigma_2) \cup \mathrm{dom}(\delta \circ \sigma_1)$

- $\nabla_2 \vdash \delta(\nabla$
- $\nabla_2 \vdash \sigma_2 \approx \delta \circ \sigma_1$

Existence of MGUs

Theorem: there is an algorithm which, given a nominal unification problem P, decides whether or not it has a solution (σ, ∇), and returns a most general one if it does.

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straightforward definition:
"iff there exists a δ such that ..."

Existence of MGUs

Theorem: there is an algorithm which, given a nominal unification problem P, decides whether or not it has a solution (σ, ∇), and returns a most general one if it does.

Proof: one can reduce all the equations to 'solved form' first (creating a substitution), and then solve the freshness problems (easy).

Reductions

A set of ($t \approx ?\ t'$) and ($a \ \#?\ t$) problems can be reduced by

$$\stackrel{\sigma}{\Longrightarrow} \quad \text{or} \quad \stackrel{\nabla}{\Longrightarrow}$$

Reductions

- $\{a.t \approx ?\ b.t'\} \uplus P \overset{\varepsilon}{\Longrightarrow}$ if $a \neq b$
  $\{t \approx ?\ (a\,b)\bullet t', a \,\#?\ t'\} \cup P$

Reductions

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  $\{t \approx ?\ (a\,b) \bullet t', a \ \#?\ t'\} \cup P$
- $\{a.t \approx ?\ a.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow} \{t \approx ?\ t'\} \cup P$

Reductions

- $\{a.t \approx? b.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$ if $a \neq b$
  $\{t \approx? (a\,b)\bullet t', a \,\#?\, t'\} \cup P$
- $\{a.t \approx? a.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow} \{t \approx? t'\} \cup P$
- $\{\pi \bullet X \approx? \pi' \bullet X\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$
  $\{a \,\#?\, X | a \in ds(\pi, \pi')\} \cup P$

Reductions

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  $\{t \approx? (a\,b)\bullet t', a \#? t'\} \cup P$
- $\{a.t \approx? a.t'\} \uplus P \overset{\varepsilon}{\Longrightarrow} \{t \approx? t'\} \cup P$
- $\{\pi\bullet X \approx? \pi'\bullet X\} \uplus P \overset{\varepsilon}{\Longrightarrow}$
  $\{a \#? X | a \in ds(\pi, \pi')\} \cup P$
- $\{\pi\bullet X \approx? t\} \uplus P \overset{\sigma}{\Longrightarrow} \sigma P$
  if X does not occur in t

Reductions

- $\{a.t \approx ?\ b.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$ if $a \neq b$
  $\{t \approx ?\ (a\,b)\bullet t', a \# ?\ t'\} \cup P$
- $\{a.t \approx ?\ a.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow} \{t \approx ?\ t'\} \cup P$
- $\{\pi \bullet X \approx ?\ \pi' \bullet X$ $\sigma = [X := \pi^{-1} \bullet t]$
  $\{a \# ?\ X \,|\, a \in ds(\pi, \pi')\} \cup P$
- $\{\pi \bullet X \approx ?\ t\} \uplus P \stackrel{\sigma}{\Longrightarrow} \sigma P$
  if X does not occur in t

Reductions

- $\{a.t \approx ?\ b.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$ if $a \neq b$
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- $\{\pi\bullet X \approx ?\ \pi'\bullet X\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$
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- $\{\pi\bullet X \approx ?\ t\} \uplus P \stackrel{\sigma}{\Longrightarrow} \sigma P$
  if X does not occur in t
- $\{a \#?\ b.t\} \uplus P \stackrel{\varnothing}{\Longrightarrow} \{a \#?\ t\} \cup P$ if $a \neq b$

Reductions

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- $\{\pi \bullet X \approx ?\ t\} \uplus P \stackrel{\sigma}{\Longrightarrow} \sigma P$
  if X does not occur in t
- $\{a \#?\ b.t\} \uplus P \stackrel{\varnothing}{\Longrightarrow} \{a \#?\ t\} \cup P$ if $a \neq b$
- $\{a \#?\ \pi \bullet X\} \uplus P \stackrel{\nabla}{\Longrightarrow} P$

Reductions

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- $\{a.t \approx ?\ a.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow} \{t \approx ?\ t'\} \cup P$
- $\{\pi \bullet X \approx ?\ \pi' \bullet X\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$
  $\{a \#?\ X | a \in ds(\pi, \pi')\} \cup P$
- $\{\pi \bullet X \approx ?\ t\} \uplus$

$\nabla = \{\pi^{-1} \bullet a \# X\}$

- $\{a \#?\ b.t\} \uplus P \stackrel{\varnothing}{\Longrightarrow} \{a \#?\ t\} \cup P$ if $a \neq b$
- $\{a \#?\ \pi \bullet X\} \uplus P \stackrel{\nabla}{\Longrightarrow} P$

Reductions

A set of $(t \approx ?\ t')$ and $(a \ \#?\ t)$ problems can be reduced by

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Reductions

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If there is a reduction

$$P \stackrel{\sigma_1}{\Longrightarrow} \ldots \stackrel{\sigma_n}{\Longrightarrow} P' \stackrel{\nabla_1}{\Longrightarrow} \ldots \stackrel{\nabla_m}{\Longrightarrow} \varnothing$$

then

$$(\sigma_n \circ \ldots \circ \sigma_1, \nabla_1 \cup \ldots \cup \nabla_m)$$

is a most general unifier for P.

Remember the Task?

Assuming that a and b are distinct variables, is it possible to find λ-terms M_1 to M_7 that make the following pairs α-equivalent?

- $\lambda a.\lambda b.(M_1\, b)$ and $\lambda b.\lambda a.(a\, M_1)$
- $\lambda a.\lambda b.(M_2\, b)$ and $\lambda b.\lambda a.(a\, M_3)$
- $\lambda a.\lambda b.(b\, M_4)$ and $\lambda b.\lambda a.(a\, M_5)$
- $\lambda a.\lambda b.(b\, M_6)$ and $\lambda a.\lambda a.(a\, M_7)$

If there is one solution for a pair, can you describe all its solutions?

Answers

$\boldsymbol{\lambda} a.\boldsymbol{\lambda} b.(M_1\, b)$ and $\boldsymbol{\lambda} b.\boldsymbol{\lambda} a.(a\, M_1)$

Answers

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

Answers

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx? (a\,b)\bullet a.\langle a, M_1\rangle\;,\; a \,\#?\, a.\langle a, M_1\rangle$$

Answers

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx? b.\langle b, (a\,b)\cdot M_1\rangle \;,\; a \,\#?\, a.\langle a, M_1\rangle$$

Answers

$$a.b.\langle M_1, b\rangle \ \approx? \ b.a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx? b.\langle b, (a\,b){\cdot}M_1\rangle \ , \ a \ \#? \ a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} \langle M_1, b\rangle \approx? \langle b, (a\,b){\cdot}M_1\rangle \ , \ a \ \#? \ a.\langle a, M_1\rangle$$

Answers

$$a.b.\langle M_1, b\rangle \ \approx ?\ \ b.a.\langle a, M_1\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx ?\ b.\langle b, (a\,b)\cdot M_1\rangle\ ,\ a \ \#?\ a.\langle a, M_1\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} \langle M_1, b\rangle \approx ?\ \langle b, (a\,b)\cdot M_1\rangle\ ,\ a \ \#?\ a.\langle a, M_1\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} M_1 \approx ?\ b\ ,\ b \approx ?\ (a\,b)\cdot M_1\ ,\ a \ \#?\ a.\langle a, M_1\rangle$$

Answers

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

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$$\stackrel{\varepsilon}{\Longrightarrow} \langle M_1, b\rangle \approx? \langle b, (a\,b)\cdot M_1\rangle\;,\; a \#? a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} M_1 \approx? b\;,\; b \approx? (a\,b)\cdot M_1\;,\; a \#? a.\langle a, M_1\rangle$$

$$\stackrel{[M_1:=b]}{\Longrightarrow} b \approx? (a\,b)\bullet b\;,\; a \#? a.\langle a, b\rangle$$

Answers

$$a.b.\langle M_1, b\rangle \ \approx ? \ b.a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx ? \ b.\langle b, (a\,b)\cdot M_1\rangle \ , \ a \ \#? \ a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} \langle M_1, b\rangle \approx ? \ \langle b, (a\,b)\cdot M_1\rangle \ , \ a \ \#? \ a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} M_1 \approx ? \ b \ , \ b \approx ? \ (a\,b)\cdot M_1 \ , \ a \ \#? \ a.\langle a, M_1\rangle$$

$$\stackrel{[M_1:=b]}{\Longrightarrow} b \approx ? \ a \ , \ a \ \#? \ a.\langle a, b\rangle$$

Answers

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx? b.\langle b, (a\,b)\cdot M_1\rangle\;,\; a \,\#?\, a.\langle a, M_1\rangle$$

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$$\stackrel{[M_1:=b]}{\Longrightarrow} b \approx? a\;,\; a \,\#?\, a.\langle a, b\rangle$$

$$\Longrightarrow FAIL$$

Answers

$$a.b.\langle M_1, b\rangle \approx ? \; b.a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx ? \; b.\langle b, (a\,b)\cdot M_1\rangle \;,\; a \, \#? \; a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} \langle M_1, b\rangle \approx ? \; \langle b, (a\,b)\cdot M_1\rangle \;,\; a \, \#? \; a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} M_1 \approx ? \; b \;,\; b \approx ? \; (a\,b)\cdot M_1 \;,\; a \, \#? \; a.\langle a, M_1\rangle$$

$$\stackrel{[M_1:=b]}{\Longrightarrow} b \approx ? \; a \;,\; a \, \#? \; a.\langle a, b\rangle$$

$$\Longrightarrow FAIL$$

$\lambda a.\lambda b.(M_1\, b) =_\alpha \lambda b.\lambda a.(a\, M_1)$ has no solution

Answers

$\lambda a.\lambda b.(b\,M_6)$ and $\lambda a.\lambda a.(a\,M_7)$

Answers

$$a.b.\langle b, M_6\rangle \;\; \approx? \;\; a.a.\langle a, M_7\rangle$$

Answers

$$a.b.\langle b, M_6\rangle \;\approx?\; a.a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle b, M_6\rangle \approx? a.\langle a, M_7\rangle$$

Answers

$$a.b.\langle b, M_6\rangle \approx ? \; a.a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle b, M_6\rangle \approx ? \; a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} \langle b, M_6\rangle \approx ? \; \langle b, (b\,a)\cdot M_7\rangle \;,\; b \,\#?\, \langle a, M_7\rangle$$

Answers

$$a.b.\langle b, M_6\rangle \;\approx?\; a.a.\langle a, M_7\rangle$$

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$$\stackrel{\varepsilon}{\Longrightarrow} b \approx? b \;,\; M_6 \approx? (b\,a){\cdot}M_7 \;,\; b \mathrel{\#?} \langle a, M_7\rangle$$

Answers

$$a.b.\langle b, M_6\rangle \;\approx?\; a.a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle b, M_6\rangle \approx? a.\langle a, M_7\rangle$$

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Answers

$$a.b.\langle b, M_6\rangle \;\approx?\; a.a.\langle a, M_7\rangle$$

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$$\stackrel{\varepsilon}{\Longrightarrow} M_6 \approx? (b\,a)\cdot M_7 \;,\; b \,\#?\, \langle a, M_7\rangle$$

$$\stackrel{[M_6:=(b\,a)\cdot M_7]}{\Longrightarrow} b \,\#?\, \langle a, M_7\rangle$$

Answers

$$a.b.\langle b, M_6\rangle \ \approx? \ a.a.\langle a, M_7\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} b.\langle b, M_6\rangle \approx? a.\langle a, M_7\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} \langle b, M_6\rangle \approx? \langle b, (b\,a)\cdot M_7\rangle \ , \ b \mathrel{\#?} \langle a, M_7\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} b \approx? b \ , \ M_6 \approx? (b\,a)\cdot M_7 \ , \ b \mathrel{\#?} \langle a, M_7\rangle$$

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$$\overset{[M_6:=(b\,a)\cdot M_7]}{\Longrightarrow} b \mathrel{\#?} \langle a, M_7\rangle$$

$$\overset{\varnothing}{\Longrightarrow} b \mathrel{\#?} a \ , \ b \mathrel{\#?} M_7$$

Answers

$$a.b.\langle b, M_6\rangle \ \approx ?\ \ a.a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle b, M_6\rangle \approx ?\ a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} \langle b, M_6\rangle \approx ?\ \langle b, (b\,a)\cdot M_7\rangle\ ,\ b \ \#?\ \langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b \approx ?\ b\ ,\ M_6 \approx ?\ (b\,a)\cdot M_7\ ,\ b\ \#?\ \langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} M_6 \approx ?\ (b\,a)\cdot M_7\ ,\ b\ \#?\ \langle a, M_7\rangle$$

$$\stackrel{[M_6:=(b\,a)\cdot M_7]}{\Longrightarrow} b\ \#?\ \langle a, M_7\rangle$$

$$\stackrel{\varnothing}{\Longrightarrow} b\ \#?\ a\ ,\ b\ \#?\ M_7$$

$$\stackrel{\varnothing}{\Longrightarrow} b\ \#?\ M_7$$

Answers

$$a.b.\langle b, M_6\rangle \;\approx?\; a.a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle b, M_6\rangle \approx? a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} \langle b, M_6\rangle \approx? \langle b, (b\,a)\cdot M_7\rangle \;,\; b \#? \langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b \approx? b \;,\; M_6 \approx? (b\,a)\cdot M_7 \;,\; b \#? \langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} M_6 \approx? (b\,a)\cdot M_7 \;,\; b \#? \langle a, M_7\rangle$$

$$\stackrel{[M_6:=(b\,a)\cdot M_7]}{\Longrightarrow} b \#? \langle a, M_7\rangle$$

$$\stackrel{\varnothing}{\Longrightarrow} b \#? a \;,\; b \#? M_7$$

$$\stackrel{\varnothing}{\Longrightarrow} b \#? M_7$$

$$\stackrel{\{b\#M_7\}}{\Longrightarrow} \varnothing$$

Answers

$$a.b.\langle b, M_6\rangle \approx ? \; a.a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle b, M_6\rangle \approx ? \; a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} \langle b, M_6\rangle \approx ?$$

$$\stackrel{\varepsilon}{\Longrightarrow} b \approx ? \; b \;,\; M$$

$$\stackrel{\varepsilon}{\Longrightarrow} M_6 \approx ? \; (b$$

$$\stackrel{[M_6:=(b\,a)\cdot M_7]}{\Longrightarrow} \; b \; \#$$

$$\stackrel{\varnothing}{\Longrightarrow} b \; \#? \; a \;,\; b \; \#? \; M_7$$

$$\stackrel{\varnothing}{\Longrightarrow} b \; \#? \; M_7$$

$$\stackrel{\{b \# M_7\}}{\Longrightarrow} \; \varnothing$$

$$\lambda a.\lambda b.(b\, M_6) \; =_\alpha \; \lambda a.\lambda a.(a\, M_7)$$

we can take M_7 to be any λ-term that does not contain free occurrences of b, so long as we take M_6 to be the result of swapping all occurrences of b and a throughout M_7